Embedded curves

$X \underset{i}{\subseteq} \mathbb{P}^n$ nonsingular projective curve (connected)

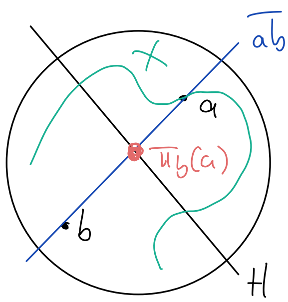
Def $b \in \mathbb{P}^n \setminus X$

$\mathbb{P}^{n-1} \cong H \subset \mathbb{P}^n \setminus b$ hyperplane

The projection from b to H is the morphism

$$\pi_b := \pi_{b,H} : \mathbb{P}^n \setminus b \longrightarrow \mathbb{P}^{n-1}$$

$$a \longmapsto \overline{ab} \cap H$$



e.g. $b = [b_0 : \cdots : b_n]$

if $b_i \neq 0$, $H = \{x_i \neq 0\}$

$$\pi_b([a_0 : \cdots : a_n]) = [b_i a_0 - a_i b_0 : \cdots : b_i a_n - a_i b_n]$$

and is given by the data

$$(\mathcal{O}_{\mathbb{P}^n \setminus b}(1), b_i x_0 - b_0 x_i, \dots, b_i x_n - b_n x_n)$$

in the linear system

$$\delta_{b,H} := \operatorname{span}\underbrace{\{b_i x_0 - b_0 x_i, \dots, b_i x_n - b_n x_n\}}_{n \text{ dimensional.}} \subset H^0(\mathcal{O}(1))$$

Def The tangent line of $a \in X \subset \mathbb{P}^n$ the unique line $L \subset \mathbb{P}^n$ such that $\mathrm{mult}_a(L \cap X)$

If X is given by the equations $\mathbb{V}(f_1, \dots, f_r)$, then $T_a(X)$ is given by the vanishing locus of the homogeneous equations:

$$\left\{ \frac{\partial f_j}{\partial x_0}(a)(a_i x_0 - a_0 x_i) + \dots + \frac{\partial f_j}{\partial x_n}(a)(a_i x_0 - a_0 x_i) \;\middle|\; j = 1, \dots, r \right\}$$

Prop $b \in \mathbb{P}^n \setminus X$

$\varphi_b := \pi_b \circ i : X \to \mathbb{P}^{n-1}$, $p \neq q \in X$

obvious from the picture.

a) $\varphi_b(p) \neq \varphi_b(q)$ iff $b \notin \overline{pq}$

b) $d_p \varphi_b$ injective iff $b \notin T_p(X)$

Thus φ_b is a closed embedding iff $b \notin \overline{pq}, T_p(X)$ for all $p, q \in X$.

Proof a) φ_b is given by linear system of hyperplanes through b

would be nice to explain this.

$\varphi_b(p) \neq \varphi_b(q)$

$\iff \exists$ a hyperplane H s.t. $p, b \in H$ but $q \notin H$

$\iff b \notin \overline{pq}$

b) $T_{X,p} \overset{d_p i}{\subseteq} T_{\mathbb{P}^n, p} \xrightarrow[\text{projection}]{d\pi_p} T_{\mathbb{P}^{n-1}, p}$ $\square$

$$\mathrm{Sec}(X) := \bigcup_{\substack{p,q \in X \\ p \neq q}} \overline{pq} \subseteq \mathbb{P}^n$$ "<u>secant variety</u>"

locally the image of

$$((X \times X) \setminus \Delta) \times \mathbb{P}^1 \longrightarrow \mathbb{P}^n$$

$$(p, q, t) \longmapsto \text{point } t \text{ on } \overline{pq}$$

$\Rightarrow$ $\mathrm{Sec}(X) \subseteq \mathbb{P}^n$ locally closed subset of dim ≤ 3

$$\mathrm{Tan}(X) = \bigcup_{p \in X} T_p(X)$$ "tangent variety"

locally the image of

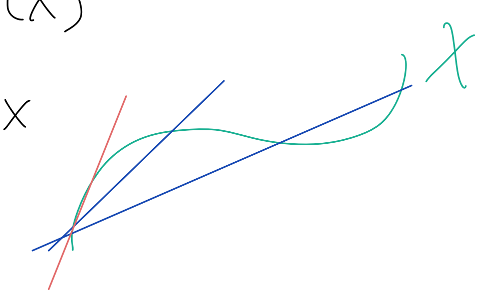
$$X \times \mathbb{P}^1 \longrightarrow \mathbb{P}^n$$

$$(p, t) \longmapsto \text{point } t \text{ on } T_p X.$$

$\Rightarrow$ $\mathrm{Tan}(X) \subseteq \mathbb{P}^n$ closed subset of dim ≤ 2.

<u>Corollary</u> φ_b is a closed embedding iff $b \notin \mathrm{Sec}(X) \cup \mathrm{Tan}(X)$

<u>Corollary</u> if $n \geq 4$ $\exists b \in \mathbb{P}^n \setminus X$ s.t. φ_b is a closed embedding.



<u>Pf</u> $\dim(\mathbb{P}^n) \geq 4 > \dim(\mathrm{Sec}(X) \cup \mathrm{Tan}(X))$

$\Rightarrow$ $\mathbb{P}^n \setminus (\mathrm{Sec}(X) \cup \mathrm{Tan}(X))$ nonempty. $\square$

Theorem Every ns proper curve X can be embedded in $\mathbb{P}^3$

Proof $\exists$ v. ample line bundles on X
$\Rightarrow \exists$ embedding $X \subset \mathbb{P}^n$
if $n \geq 3$ done
else $\exists\, b \in \mathbb{P}^n \setminus (\mathrm{Sec}(X) \cup \mathrm{Tan}(X))$
$\Rightarrow X \subset \mathbb{P}^{n-1}$
Conclude by induction. $\square$

I want to mention the following theorem:

Thm If $g(X) \geq 2$, then $\mathrm{Aut}(X)$ is finite.

Proof idea:

if X hyperelliptic then every automorphism permutes the ramification points of the hyperelliptic cover.

if X not hyperelliptic, then $\mathrm{Aut}(X)$ permutes the hyperosculation points of the canonical embedding $\square$.

Bertini theorem

<u>Def</u> Consider the projective space $\mathbb{P}^n$ w/ coordinates $x_0, \dots, x_n$

The <u>dual projective space</u> is a projective space $(\mathbb{P}^n)^\vee$ w/ coordinates $a_0 \cdots a_n$ together with the data of the <u>incidence variety</u>:

$$I = \{a_0x_0 + \cdots + a_nx_n = 0\} = \{(p, H) \mid p \in H\}$$

$$\begin{array}{ccccc} & & \cap & & \\ \mathbb{P}^n & \xleftarrow{\pi} & \mathbb{P}^n \times (\mathbb{P}^n)^\vee & \xrightarrow{\pi^\vee} & (\mathbb{P}^n)^\vee \end{array}$$

Every point $[H] \overset{= [\alpha_0 : \cdots : \alpha_n]}{\in} (\mathbb{P}^n)^\vee$ is interpreted as a hyperplane

$$\mathbb{P}^n \supset H = \{\alpha_0x_0 + \cdots + \alpha_nx_n\} = \pi\left(I \cap (\pi^\vee)^{-1}([H])\right)$$

via this correspondence and vice versa.

<u>Def</u> $X \subseteq \mathbb{P}^n$ subvariety.

$$Z_X := \left\{(p, H) \;\middle|\; \begin{array}{l} p \in X,\ p \in H \\ \text{s.t. } p \in X \cap H \text{ singular} \\ \text{or } \quad H \subseteq X \end{array}\right\}$$

$$\underset{\text{if } X \text{ smth.}}{=} \left\{(p, H) \;\middle|\; \begin{array}{l} p \in X\ p \in H \\ T_pX \subseteq H \text{ or } H \subseteq X \end{array}\right\}$$

can be explicitly described.

in terms of equations.

$X^\vee := \pi^\vee(Z_X)$ dual variety

Theorem (Bertini)

$X \subset \mathbb{P}^n$ ns subvariety of dim d.

$\Rightarrow \exists\, U \subset (\mathbb{P}^n)^\vee \;\forall\, [H] \in U$

1) $X \not\subset H$

2) $X \cap H$ is ns. of dim $d-1$.

Proof sketch:

Can take $U := \mathbb{P}^n \setminus (X^\vee)$

fibres of $\pi : Z_X \longrightarrow X$ over p

$$= \{[H] \in (\mathbb{P}^n)^\vee \mid \underbrace{T_pX \subset H}_{d\text{-dim condition on } H},\ \underbrace{p \in H}_{1\text{-dim cond. on } H}\}$$

$$= \mathbb{P}^{n-1-\dim(X)}$$

$\Rightarrow \dim(Z_X) \leq n-1-\dim(X) + \dim(X)$

$\Rightarrow \dim(X^\vee) \leq n-1.$

$\Rightarrow U$ nonempty. $\square$

Hilbert polynomial.

$X \subset \mathbb{P}^n$ projective variety

$\mathcal{O}_X(1) := i^* \mathcal{O}_{\mathbb{P}^n}(1)$ very ample line bundle

$\mathcal{F} \in QCoh(X)$ $\quad \mathcal{F}(m) := \mathcal{F} \otimes \mathcal{O}_X(1)^{\otimes m}$

<u>Theorem</u> For all $\mathcal{F} \in Coh(X)$ there is a polynomial $P_{\mathcal{F}}(t) \in \mathbb{Q}[t]$ s.t.

$$\forall m \in \mathbb{Z}: \quad P_{\mathcal{F}}(m) = \chi(X, \mathcal{F}(m))$$

Moreover $P_{\mathcal{F}}(m) = h^0(\mathcal{F}(m))$ for $m \gg 0$

<u>Def.</u> · $P_{\mathcal{F}}(\ )$ is called the <u>Hilbert polynomial</u> of $\mathcal{F}$

· $P_X(\ ) := P_{\mathcal{O}_X}(t)$ is the <u>Hilbert poly.</u> of X

! depends on $\mathcal{O}_X(1)$ / embeddin $X \subset \mathbb{P}^n$

<u>Proof</u> replacing $\mathcal{F}$ by $i_* \mathcal{F}$ $\Rightarrow$ wlog $X = \mathbb{P}^n$

Use Hilbert syzygy theorem w/o proof:

<u>Fact</u>: For all $\mathcal{F} \in Coh(\mathbb{P}^n)$ there is a finite resolution of length $n+1$

$$0 \to \bigoplus_{j \in J_n} \mathcal{O}(m_{nj}) \to \cdots \to \bigoplus_{j \in J_0} \mathcal{O}(m_{0j}) \to \mathcal{F} \to 0$$

additivity of χ

$$\Rightarrow \chi(\mathcal{F}(m)) = \sum_{i=0}^{n} (-1)^i \sum_{j \in J_i} \chi(\mathcal{O}(m_{i,j}+m))$$

$$= \sum_{i=0}^{n} (-1)^i \sum_{j \in J_i} \binom{m_{i,j}+m+n}{n}$$

$$= \left(\frac{\prod_{k=1}^{n} m_{i,j}+t+k}{n!} \right) \Bigg|_{t=m}$$

$\square$

<u>Example</u> $P_{\mathbb{P}^n}(t) = \binom{t+n}{n}$

<u>Lemma</u> If $P(t) = \lambda_n t^n +$ lower order terms $\in \mathbb{Q}[t]$

s.t. $P(m) \in \mathbb{Z}$ for all $m \in \mathbb{Z}$.

Then $\lambda_n = \frac{d}{n!}$ for $d \in \mathbb{Z}$.

<u>Lemma</u> $\deg(P_X(t)) = \dim(X)$

<u>Def</u> The <u>degree</u> of $X \subset \mathbb{P}^n$ is

$$\deg(X) := \dim(X)!\, \lambda_{\dim}$$

$$P_X(t) = \lambda_{\dim(X)}\, t^{\dim(X)} + \dots$$

Lemma $Y_d \subseteq \mathbb{P}^n$ degree d Hypersurface. then $\deg(Y_d) = d$.

Proof Have s.e.s
$$0 \to \mathcal{O}_{\mathbb{P}^n}(m-d) \to \mathcal{O}_{\mathbb{P}^n}$$
$$\Rightarrow P_{Y_d}(m) = \binom{m+n}{n} - \binom{m-d+n}{n}$$
$$= \frac{(m+1)\cdots(m+n)}{n!} - \frac{(m-d+1)\cdots(m-d+n)}{n!}$$
$$= \frac{\left(\sum_{i=1}^{n} i\right) m^{n-1} + \text{lower order terms}}{n!}$$
$$- \sum_{i=1}^{n}(i-d)\, m^{n-1} + \text{lower order terms}$$
$$= \frac{ndm^{n-1}}{n!} + \text{lower order terms.} \quad \square$$

Lemma X curve $\mathcal{L}$ v. ample line bun.
$$X \xhookrightarrow{i} \mathbb{P}(H^0(\mathcal{L}))$$
Then $\deg(X) = \deg(\mathcal{L})$

Pf $\chi(\mathcal{L}^{\otimes m}) = 1-g+\deg(\mathcal{L})m$
by RR $\square$

<u>Remark</u> constant term

$$P_X(0) = \chi(X, \mathcal{O}_X)$$

does not depend on $X \subseteq \mathbb{P}^n$

$$p_a(X) = (-1)^{\dim(X)} (\chi(X, \mathcal{O}_X) - 1)$$

<u>arithmetic genus</u>

<u>Theorem</u> (Bezout)

$X \subset \mathbb{P}^n$ irred. variety $Y \subset \mathbb{P}^n$ hypersurf.

$$\Rightarrow \deg(Y \cap X) = \deg(Y) \deg(X)$$

<u>Pf</u> Use ses.

$$0 \to \mathcal{I}_{Y \cap X} \to \mathcal{O}_X \to \mathcal{O}_{X \cap Y} \to 0$$

$$\mathcal{I}_{Y \cap X} \cong \mathcal{O}_X(-d)$$

$\square$

Now have the right conceptual tools to make sense of sentences like:

"The tangent line of a point on a plane curve X is the unique line L s.t. it intersects X at p w/ degree ≥ 2 "